

CLASS NUMBER ONE CRITERION FOR SOME NON-NORMAL TOTALLY REAL CUBIC FIELDS

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ABSTRACT. Let $\{K_m\}_{m \geq 4}$ be the family of non-normal totally real cubic number fields defined by the irreducible cubic polynomial $f_m(x) = x^3 - mx^2 - (m+1)x - 1$, where m is an integer with $m \geq 4$. In this paper, we will give a class number one criterion for K_m .

1. INTRODUCTION

It has been known for a long time that there exists close connection between prime producing polynomials and class number one problem for some number fields. Rabinowitsch[9] proved that for a prime number q , the class number of $\mathbb{Q}(\sqrt{1-4q})$ is equal to one if and only if $k^2 + k + q$ is prime for every $k = 0, 1, \dots, q-2$. For real quadratic fields, many authors[2, 3, 8, 11] considered the connection between prime producing polynomials and class number. For the simplest cubic fields, Kim and Hwang[6] gave a class number one criterion which is related to some prime producing polynomials. The aim of this paper is to give a class number one criterion for some non-normal totally real cubic fields. Its criterion provides some polynomials having almost prime values in a given interval. The method done in this paper is basically same as one in [2, 3, 6].

Let $\zeta_K(s)$ be the Dedekind zeta function of an algebraic number field K and $\zeta_K(s, P)$ be the partial zeta function for the principal ideal class P of K . Then we have

$$\zeta_K(-1) \leq \zeta_K(-1, P).$$

Halbritter and Pohst[5] developed a method of expressing special values of the partial zeta functions of totally real cubic fields as a finite sum involving norm, trace, and 3-fold Dedekind sums. Their result has been exploited by Byeon[1] to give an explicit formula for the values of the partial zeta functions of the simplest cubic fields. Kim and

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Hwang[6] gave a class number one criterion for the simplest cubic fields by estimating the value $\zeta_K(-1)$ and combining Byeon's result. In this paper, we will do this kind of work in some non-normal totally real cubic fields. First, we apply Halbritter and Pohst's formula to our cubic fields, and then evaluate the upper bound of $\zeta_K(-1)$ by using Siegel's formula. Finally, combining this computation, we give a class number one criterion for some non-normal totally real cubic fields. Halbritter and Pohst[5] proved:

Theorem 1.1. *Let K be a totally real cubic field with discriminant Δ . For $\alpha \in K$, the conjugates are denoted by α' and α'' , respectively. Furthermore, for $\alpha \in K$, let $\text{Tr}(\alpha) := \alpha + \alpha' + \alpha''$ and $N(\alpha) := \alpha \cdot \alpha' \cdot \alpha''$. Let $\widehat{K} := K(\sqrt{\Delta})$, $k \in \mathbb{N}$, $k \geq 2$, and $\{\epsilon_1, \epsilon_2\}$ be a system of fundamental units of K . Define L by $L := \ln|\epsilon_1/\epsilon_1'| \ln|\epsilon_2'/\epsilon_2''| - \ln|\epsilon_1'/\epsilon_1''| \ln|\epsilon_2/\epsilon_2''|$. Let W be an integral ideal of K with basis $\{w_1, w_2, w_3\}$. Let $\rho = \widetilde{w}_3$ for a dual basis $\widetilde{w}_1, \widetilde{w}_2, \widetilde{w}_3$ of W subject to*

$$\text{Tr}(w_i \widetilde{w}_j) = \delta_{ij} (1 \leq i, j \leq 3).$$

For $j = 1, 2$, set

$$E_j = \begin{pmatrix} 1 & 1 & 1 \\ \epsilon_j & \epsilon_j' & \epsilon_j'' \\ \epsilon_1 \epsilon_2 & \epsilon_1' \epsilon_2' & \epsilon_1'' \epsilon_2'' \end{pmatrix}$$

and

$$B_\rho = \begin{pmatrix} \rho w_1 & \rho w_2 & \rho w_3 \\ \rho' w_1' & \rho' w_2' & \rho' w_3' \\ \rho'' w_1'' & \rho'' w_2'' & \rho'' w_3'' \end{pmatrix}.$$

For $\tau_1, \tau_2 \in K$, $\nu = 1, 2$, set

$$M(k, \nu, \tau_1, \tau_2) := 0$$

if $\det E_\nu = 0$, otherwise

$$\begin{aligned}
M(k, \nu, \tau_1, \tau_2) &:= \text{sign}(L)(-1)^\nu [\widehat{K} : \mathbb{Q}]^{-1} \frac{(2\pi i)^{3k}}{(3k)!} N(\rho)^k \\
&\cdot \sum_{m_1=0}^{3k} \sum_{m_2=0}^{3k} \binom{3k}{m_1, m_2} \\
&\cdot \left\{ \frac{\det E_\nu}{|\det(E_\nu B_\rho)|^3} \mathbf{B}(3, m_1, m_2, 3k - (m_1 + m_2), (E_\nu B_\rho)^*, \mathbf{0}) \right. \\
&\cdot \sum_{\kappa_1=0}^{k-1} \sum_{\kappa_2=0}^{k-1} \sum_{\mu_1=0}^{k-1} \sum_{\mu_2=0}^{k-1} \binom{m_1 - 1}{k - 1 - (\kappa_1 + \kappa_2), k - 1 - (\mu_1 + \mu_2)} \\
&\cdot \binom{m_2 - 1}{\kappa_1, \mu_1} \binom{3k - 1 - (m_1 + m_2)}{\kappa_2, \mu_2} \\
&\cdot \text{Tr}_{\widehat{K}/\mathbb{Q}}(\tau_1^{\kappa_1 + \kappa_2} \tau_1'{}^{\mu_1 + \mu_2} \tau_1''{}^{3k - 2 - (m_1 + \kappa_1 + \kappa_2 + \mu_1 + \mu_2)} \\
&\cdot \tau_2^{\kappa_2} \tau_2'{}^{\mu_2} \tau_2''{}^{3k - 1 - (m_1 + m_2 + \kappa_2 + \mu_2)}) \left. \right\},
\end{aligned}$$

where $(E_\nu B_\rho)^*$ denotes the transposed matrix of $(E_\nu B_\rho)$, and

$$\begin{aligned}
C(k, \nu, \tau_1, \tau_2) &:= \text{sign}(L)(-1)^{\nu+1} \frac{(2\pi i)^{3k}}{12 \cdot (3k - 2)(k - 1)!^3} N(\rho)^k \\
&\cdot \widetilde{B}_{3k-2}(0) |\det B_\rho|^{-1} \text{sign}(\det E_\nu) \\
&\cdot \{ \text{sign}((\tau_1 \tau_2 - \tau_1' \tau_2')(\tau_1 - \tau_1')) + \text{sign}((\tau_1' \tau_2' - \tau_1'' \tau_2'')(\tau_1' - \tau_1'')) \\
&+ \text{sign}((\tau_1'' \tau_2'' - \tau_1 \tau_2)(\tau_1'' - \tau_1)) + \text{sign}(\tau_1''(\tau_1 - \tau_1')(\tau_2' - \tau_2)) \\
&+ \text{sign}(\tau_1(\tau_1' - \tau_1'')(\tau_2'' - \tau_2')) + \text{sign}(\tau_1'(\tau_1'' - \tau_1)(\tau_2 - \tau_2'')) \\
&+ N(\tau_2) [\text{sign}(\tau_1''(\tau_2 - \tau_2')(\tau_1 \tau_2 - \tau_1' \tau_2')) \\
&+ \text{sign}(\tau_1(\tau_2' - \tau_2'')(\tau_1' \tau_2' - \tau_1'' \tau_2'')) + \text{sign}(\tau_1'(\tau_2'' - \tau_2)(\tau_1'' \tau_2'' - \tau_1 \tau_2))] \}.
\end{aligned}$$

Define

$$\begin{aligned}
\zeta(k, W, \epsilon_1, \epsilon_2) &:= M(k, 1, \epsilon_1, \epsilon_2) + M(k, 2, \epsilon_2, \epsilon_1) \\
&+ C(k, 1, \epsilon_1, \epsilon_2) + C(k, 2, \epsilon_2, \epsilon_1).
\end{aligned}$$

Let $\zeta_K(s, K_0)$ be the partial zeta function of an absolute ideal class K_0 of K and $W \in K_0^{-1}$. Then we have

$$(1) \quad \zeta_K(2k, K_0) = \frac{1}{2} \text{Norm}(W)^{2k} \zeta(2k, W, \epsilon_1, \epsilon_2).$$

Remark 1. For $k, l, m \in \mathbb{Z}$,

$$\binom{k}{l, m} := \begin{cases} \frac{k!}{l!m!(k-(l+m))!} & \text{if } k, l, m, k - (l + m) \in \mathbb{N} \cup \{0\} \\ (-1)^{l+m} \binom{l+m}{l} & \text{if } k = -1 \text{ and } l, m \in \mathbb{N} \cup \{0\} \\ 0 & \text{otherwise.} \end{cases}$$

Remark 2. Let $A = (a_{ij})_{n,n}$ be a regular (n, n) -matrix with integral coefficients, $(A_{ij})_{n,n} := (\det A)A^{-1}$. Let

$$\tilde{B}_r(x) := \begin{cases} B_r(x - [x]) & \text{if } r = 0 \text{ or } r \geq 2 \text{ or } r = 1 \wedge x \notin \mathbb{Z} \\ 0 & \text{if } r = 1 \wedge x \in \mathbb{Z}, \end{cases}$$

where $B_r(y)$ is defined as usual by $ze^{yz}(e^z - 1)^{-1} = \sum_{r=0}^{\infty} B_r(y)z^r/r!$. Then, for $\mathbf{r} = (r_1, \dots, r_n) \in (\mathbb{N} \cup \{0\})^n$,

$$\mathbf{B}(n, \mathbf{r}, A, \mathbf{0}) = \sum_{\kappa_1=0}^{|\det A|-1} \cdots \sum_{\kappa_n=0}^{|\det A|-1} \prod_{i=1}^n \tilde{B}_{r_i}\left(\frac{1}{\det A} \sum_{j=1}^n A_{ij}\kappa_j\right).$$

Next, we introduce Siegel's formula for the values of the Dedekind zeta function of a totally real algebraic number field at negative odd integers.

For an ideal I of the ring of integers \mathcal{O}_K , we define the sum of ideal divisors function $\sigma_r(I)$ by

$$(2) \quad \sigma_r(I) = \sum_{J|I} N_{K/\mathbb{Q}}(J)^r,$$

where J runs over all ideals of \mathcal{O}_K which divide I . Note that, if $K = \mathbb{Q}$ and $I = (n)$, our definition coincides with the usual sum of divisors function

$$(3) \quad \sigma_r(n) = \sum_{\substack{d|n \\ d>0}} d^r.$$

Now let K be a totally real algebraic number field. For $l, b = 1, 2, \dots$, we define

$$(4) \quad S_l^K(2b) = \sum_{\substack{\nu \in \mathcal{D}_K^{-1} \\ \nu \gg 0 \\ \text{Tr}_{K/\mathbb{Q}}(\nu) = l}} \sigma_{2b-1}((\nu)\mathcal{D}_K),$$

where \mathcal{D}_K is the different of K . At this moment, we remark that this is a finite sum. Siegel[10] proved:

Theorem 1.2. *Let b be a natural number, K a totally real algebraic number field of degree n , and $h = 2bn$. Then*

$$(5) \quad \zeta_K(1 - 2b) = 2^n \sum_{l=1}^r b_l(h) S_l^K(2b).$$

The numbers $r \geq 1$ and $b_1(h), \dots, b_r(h) \in \mathbb{Q}$ depend only on h . In particular,

$$(6) \quad r = \dim_{\mathbb{C}} \mathcal{M}_h,$$

where \mathcal{M}_h denotes the space of modular forms of weight h . Thus by a well-known formula,

$$r = \begin{cases} \left[\frac{h}{12} \right] & \text{if } h \equiv 2 \pmod{12} \\ \left[\frac{h}{12} \right] + 1 & \text{if } h \not\equiv 2 \pmod{12}. \end{cases}$$

Now, we will introduce our target fields. Let $m(\geq 4)$ be a rational integer and K_m (or simply K) = $\mathbb{Q}(\alpha)$ be the non-normal totally real cubic number field (whose arithmetic was studied in [7]) associated with the irreducible cubic polynomial

$$(7) \quad f_m(x) = x^3 - mx^2 - (m+1)x - 1 \in \mathbb{Z}[x]$$

of positive discriminant

$$D_m = (m^2 + m - 3)^2 - 32 > 0$$

and with three distinct real roots $\alpha_3 < \alpha_2 < \alpha_1 = \alpha$. We borrow known results for arithmetic of K_m .

Theorem 1.3. (1) *The set $\{1, \alpha, \alpha^2\}$ forms an integral basis of the ring \mathcal{O}_K of algebraic integers of K if and only if one of the following conditions holds true:*

- (i) $m \not\equiv 3 \pmod{7}$ and D_m is square-free,
 - (ii) $m \equiv 3 \pmod{7}$, $m \not\equiv 24 \pmod{7^2}$ and $\frac{D_m}{7^2}$ is square-free.
- (2) *The full group of algebraic units of K_m is $\langle -1, \alpha, \alpha + 1 \rangle$.*

Proof. See [7]. □

2. CLASS NUMBER ONE CRITERION FOR K_m

In this section, to have the value of $\zeta_K(-1, P)$, we apply Theorem 1.1 to K_m . On the other hand, we evaluate the upper bound of $\zeta_K(-1)$ by using Theorem 1.2. Finally, combining these results, we give a class number one criterion for K_m .

We take $W = \mathcal{O}_K = (\alpha)$. Since the ideal class containing \mathcal{O}_K is the principal ideal class P , by (1), we have

$$\zeta_K(2, P) = \frac{1}{2} \zeta(2, \mathcal{O}_K, \alpha, \alpha + 1).$$

By definition,

$$\begin{aligned} \zeta(2, \mathcal{O}_K, \alpha, \alpha + 1) &= M(2, 1, \alpha, \alpha + 1) + M(2, 2, \alpha + 1, \alpha) \\ &\quad + C(2, 1, \alpha, \alpha + 1) + C(2, 2, \alpha + 1, \alpha). \end{aligned}$$

Let $\{\widetilde{w}_1, \widetilde{w}_2, \widetilde{w}_3\}$ be a dual basis of \mathcal{O}_K . Then, by a simple computation, we get

$$\begin{aligned} \rho = \widetilde{w}_3 &= \frac{-1}{D_m} \{ (m^3 + 5m^2 + 5m + 4) + (2m^3 + 7m^2 + 7m + 9)\alpha \\ &\quad - 2(m^2 + 3m + 3)\alpha^2 \} \end{aligned}$$

This makes it possible to determine matrices E_1, E_2 and B_ρ . Now, we note that 3-fold Dedekind sum $\mathbf{B}(3, m_1, m_2, 6 - (m_1 + m_2), (E_\nu B_\rho)^*, \mathbf{0})$ vanishes when m_1 or m_2 is odd. Next, we need the computation for trace. This computation is very long but elementary. Combining these data, we have

$$\begin{aligned} M(2, 1, \alpha, \alpha + 1) &= -(4m^9 + 54m^8 + 304m^7 + 979m^6 \\ &\quad + 2119m^5 + 3234m^4 + 3327m^3 + 2067m^2 + 72m - 714)\pi^6/2835D_m^{3/2} \end{aligned}$$

$$\begin{aligned} M(2, 2, \alpha + 1, \alpha) &= (4m^9 + 54m^8 + 304m^7 + 985m^6 \\ &\quad + 2137m^5 + 3204m^4 + 3237m^3 + 2091m^2 + 144m - 714)\pi^6/2835D_m^{3/2} \end{aligned}$$

On the other hand, the calculation of $C(2, 1, \alpha, \alpha + 1)$ (resp. $C(2, 2, \alpha + 1, \alpha)$) is simpler than one of $M(2, 1, \alpha, \alpha + 1)$ (resp. $M(2, 2, \alpha + 1, \alpha)$). In fact,

$$C(2, 1, \alpha, \alpha + 1) = \frac{2\pi^6}{45D_m^{3/2}}, \quad C(2, 2, \alpha + 1, \alpha) = -\frac{2\pi^6}{45D_m^{3/2}}.$$

Then, by collecting these results, we have the following theorem.

Theorem 2.1. *Let $m(\geq 4)$ be an integer which satisfies the conditions of Theorem 1.3 and K_m the non-normal totally real cubic field defined by (7). Let P be the principal ideal class of K_m . Then we have*

$$\zeta_K(2, P) = \frac{m(m^5 + 3m^4 - 5m^3 - 15m^2 + 4m + 12)\pi^6}{945(D_m)^{3/2}}.$$

Moreover, by a functional equation,

$$\zeta_K(-1, P) = -\frac{m(m^5 + 3m^4 - 5m^3 - 15m^2 + 4m + 12)}{7560}.$$

Next, by Theorem 1.2, noting that $b_1(8) = -1/504$ (cf. [12]), we have

$$\zeta_K(-1) = -\frac{8}{504}S_1^K(2) = -\frac{8}{504} \sum_{\nu \in S_1} \sigma_1((\nu)\mathcal{D}_K),$$

where

$$S_1 := \{\nu \in K \mid \nu \in \mathcal{D}_K^{-1}, \nu \gg 0, \text{Tr}_{K/\mathbb{Q}}(\nu) = 1\}.$$

Let T be the set of integral points in (s, t) -plane corresponding to S_1 by one-to-one correspondence in [4, Proposition 2.1]. This set has been completely determined in [4, Theorem 2.3] as follows:

$$\begin{aligned} T = \{ & (1, 1), \quad (1, 2), \quad \dots, \quad (1, m-1), \\ & (2, 2), \quad (2, 3), \quad \dots, \quad (2, m), \\ & (3, 3), \quad (3, 4), \quad \dots, \quad (3, m), \\ & \dots\dots\dots \dots\dots\dots \dots\dots\dots, \\ & (m-2, m-2), (m-2, m-1), (m-2, m), \\ & (m-1, m) \}. \end{aligned}$$

Furthermore, by (26) of [4]

$$N((\nu)\mathcal{D}_K) = |f_m(s, t)|,$$

where

$$\begin{aligned} f_m(s, t) = & (-s^2 + (t+1)s)m^2 + ((t-2)s^2 - (t^2-t)s - (t^2+t))m \\ & + (s^3 - 2s^2 - (t^2-3t-1)s + t^3 - t - 1). \end{aligned}$$

One can easily check that $f_m(s, t) > 1$ for all $(s, t) \in T$. Therefore, we have the following inequalities

$$\begin{aligned}
(8) \quad \zeta_K(-1) &\leq -\frac{8}{504} \sum_{\nu \in S_1} (1 + N((\nu)\mathcal{D}_K)) \\
&= -\frac{8}{504} \{ \#S_1 + \sum_{\nu \in S_1} N((\nu)\mathcal{D}_K) \} \\
&= -\frac{8}{504} \left\{ \frac{1}{2}(m^2 + m - 6) + \sum_{(s,t) \in T} f_m(s, t) \right\} \\
&= -\frac{8}{504} \left\{ \frac{1}{2}(m^2 + m - 6) + \sum_{t=1}^{m-1} f_m(1, t) \right. \\
&\quad \left. + \sum_{s=2}^{m-2} \sum_{t=s}^m f_m(s, t) + f_m(m-1, m) \right\} \\
&= -\frac{m(m^5 + 3m^4 - 5m^3 - 15m^2 + 4m + 12)}{7560} \\
&= \zeta_K(-1, P),
\end{aligned}$$

and equality holds in (8) if and only if $(\nu)\mathcal{D}_K$ is a prime ideal for all $\nu \in S_1$. Combining this computation, we give a class number one criterion for K_m .

Theorem 2.2. *Let $m(\geq 4)$ be an integer which satisfies the conditions of Theorem 1.3 and K_m the non-normal totally real cubic field defined by (7). Then we have*

$$h_K = 1 \text{ if and only if } (\nu)\mathcal{D}_K \text{ is a prime ideal for all } \nu \in S_1.$$

On the other hand, Louboutin[7] showed:

$$m = 4, 5, 6, 8 \text{ gives all the values of } m \text{ such that } h_K = 1.$$

Therefore, we can conclude that

$$m = 4, 5, 6, 8 \text{ if and only if } (\nu)\mathcal{D}_K \text{ is a prime ideal for all } \nu \in S_1.$$

Remark 3. Unlike in the simplest cubic fields which is a Galois extension of \mathbb{Q} , $N((\nu)\mathcal{D}_K) = |f_m(s, t)|$ is necessarily not prime where $(\nu)\mathcal{D}_K$ is a prime ideal for each $\nu \in S_1$. For example, $f_m(2, 3) = (2m - 5)^2$ for the point $(2, 3)$ in T , but if $m = 4, 5, 6, 8$, then each integral ideal $(\nu)\mathcal{D}_K$ corresponding to the point $(2, 3)$ is prime. Furthermore, $f_m(s, t)$ is a prime for all points only except $(2, 3)$ in T when $m = 4, 5, 6, 8$.

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